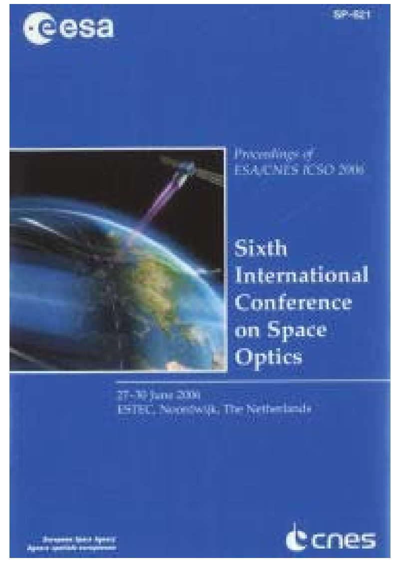
International Conference on Space Optics—ICSO 2006

Noordwijk, Netherlands

27–30 June 2006

Edited by Errico Armandillo, Josiane Costeraste, and Nikos Karafolas



Description of quantum states using in free space optic communication

Petr Kučera

International Conference on Space Optics — ICSO 2006, edited by Errico Armandillo, Josiane Costeraste,
Nikos Karafolas, Proc. of SPIE Vol. 10567, 105673V · © 2006 ESA and CNES
CCC code: 0277-786X/17/$18 · doi: 10.1117/12.2308115

DESCRIPTION OF QUANTUM STATES USING IN FREE SPACE OPTIC COMMUNICATION

Petr Kučera[(1)]

[(1)] *Institute of Radio Electronics, Faculty of Electrical Engineering and Communication Brno, University of Technology, Czech Republic , 612 00 Brno, Purkyňova 118*
Email: xkucer24@stud.feec.vutbr.cz

ABSTRACT

In the article we concentrate our attention on the quantum description of states which are prepared by light sources. The main goal of the article is the determination of density matrix of background radiation source. It is shown that these matrix elements satisfy Geometric distribution in the number state representation.

1. INTRODUCTION

Quantum information theory is the integration of quantum theory into classical information theory. The most dramatic applications of quantum information theory are used in new technologies like ultimate safe cryptography (an uncertainty of the overlap between two different basis is used) and quantum computers. These technologies are based on the quantity called quantum information which is quite different from the information in classical sense.

In the article we are interested in quantum optical communication. Hence the information is carried with using photons (generally with states of systems). In the communication structure there are two the most important phases. The first phase is the quantum state preparation (this phase is represented by light sources). The second phase, analogous to the first one, is the measuring process (light detector).

Firstly we mention about some notation used in quantum mechanics. A quantum system is whatever admits a closed dynamical description within quantum theory. A state (or quantum signal) is characterized by the probabilities of the various outcomes of every conceivable test. Mathematically the states of systems are represented by vectors and every physical process is completely determined by particular matrix. To specify the matrix, first there must be chosen a basis (roughly speaking basis states are states which are absolutely different). Thus matrix can be expressed in various forms depending on the choice of a basis. Quantum systems can manipulate two types of states. Pure states are represented by vectors in Hilbert space and mixed states are the statistical mixture of pure states. Both type of states can by completely described by density matrices.

2. SINGLE PHOTON SOURCE

If the wave function of a photon can be factorized than one can write

$$\psi = \psi_P \otimes \psi_D \otimes \psi_M , \tag{1}$$

where

$$\psi_P = \sum_{i=1}^{2} p_i \left| p_i \right\rangle \tag{2}$$

$$\psi_D = \sum_{\text{directions}} d_j \left| d_j \right\rangle, \psi_M = \sum_{\text{momentum}} m_k \left| m_k \right\rangle. \tag{3}$$

Hence the components of tensor product (1) describe separately polarization, position and momentum of a photon. Note that it is not always necessary consider all degrees of freedom. According to the superposition principle one can Eq. (1) rewrite into the form

$$\left| \varphi \right\rangle = \sum_{i,j,k} a_{ijk} \left| p_i \right\rangle \otimes \left| d_j \right\rangle \otimes \left| m_k \right\rangle \tag{4}$$

and normalization condition is hold

$$\sum_{ijk} \left| a_{ijk} \right|^2 = 1, \tag{5}$$

where single index $ijk = 1, 2, .., i \cdot j \cdot k$. Complex coefficients a_{ijk} specify the single photon source and are determined by measuring processes. Thus Eq. (4) has meaning only if there is an experimental process which is able to distinguish the basis in Eq. (4). Such measuring process can be constructed by using calcite crystals, dispersion prisms and photodiodes.

Let us show how to determine density matrix of single photon source. For simplicity suppose that ψ_D, ψ_M in Eq. (1) vanish. Then every density matrix of order two has following form

$$\boldsymbol{\rho} = d \cdot \mathbf{1} + a \cdot \boldsymbol{\sigma}_x + b \cdot \boldsymbol{\sigma}_y + c \cdot \boldsymbol{\sigma}_z , \tag{6}$$

where $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y, \boldsymbol{\sigma}_z$ are Pauli matrices and d, a, b, c are real coefficients. With using the trace condition $\mathrm{Tr}(\boldsymbol{\rho}) = 1$ Eq. (6) can be written as

Proc. '6th Internat. Conf. on Space Optics', ESTEC, Noordwijk, The Netherlands, 27-30 June 2006 (ESA SP-621, June 2006)

$$\boldsymbol{\rho} = \frac{1}{2}\left(\mathbf{1} + a \cdot \boldsymbol{\sigma}_x + b \cdot \boldsymbol{\sigma}_y + c \cdot \boldsymbol{\sigma}_z\right) \qquad (7)$$

For instance matrix elements ρ_{12}, ρ_{21} of $\boldsymbol{\rho}$ are given by the following formulas

$$\rho_{12} = (a - \mathrm{i}b)/2, \; \rho_{21} = (a + \mathrm{i}b)/2. \qquad (8)$$

Hence

$$b = \mathrm{i}(\rho_{12} - \rho_{21}) = \mathrm{Tr}(\boldsymbol{\rho} \cdot \boldsymbol{\sigma}_y) = \langle \boldsymbol{\sigma}_y \rangle. \qquad (9)$$

One can see that coefficient b is obtained by measuring of the observable $\boldsymbol{\sigma}_y$. In the similar way the coefficients a, c are obtained.

3. ENTEGLEMENT

The concept of information entanglement is widely used in quantum cryptography and quantum teleportation. Two systems are said to be entangled if their wave function can not be factorized. Hence the superposition principle has stronger (fundamental) meaning than the ordinary tensor product. A general wave function for N entangled photons is [1]

$$\psi = f(\mathbf{r}_1, \mathbf{r}_2 ..., \mathbf{r}_n)(\mathbf{u}_1\mathbf{u}_2 ... \mathbf{u}_N + \mathbf{v}_1\mathbf{v}_2 ... \mathbf{v}_N), \qquad (10)$$

where $\mathbf{u}, \mathbf{v}$ are the eigenvectors of one of Pauli matrix (commonly $\boldsymbol{\sigma}_z$ is chosen) and the function f ensures that photons are spatially separated.

Let us suppose two systems A and B. Their states can be defined as

$$|\psi\rangle_A = \sum_{i=1}^{N} a_i |a_i\rangle, \; \psi_B = \sum_{j=1}^{N} b_j |b_j\rangle. \qquad (11)$$

If these systems do not interact with each other, one can write for their joint state

$$|\psi\rangle_{AB} = |\varphi\rangle_A \otimes |\phi\rangle_B = \sum_{i,j} a_i b_j |a_i\rangle \otimes |b_j\rangle. \qquad (12)$$

On the other hand according to the superposition principle we have

$$|\varphi\rangle_{AB} = \sum_{i,j} c_{ij} |a_i\rangle \otimes |b_j\rangle. \qquad (13)$$

If $c_{ij} \neq a_i b_j$, then the systems are entangled (the systems mutually interact).

Now we show whether it is possible to identify Eq. (12) and Eq. (13). Assume that in the state $|\varphi\rangle_{AB}$ the coefficient c_{mn} vanishes. If we identify Eq. (12) and Eq.(13) so we can write $c_{mn} = a_m b_n$. Hence one coefficient of a_m, b_n must vanish (recall that $c_{mn} = 0$). Then one term drops out of Eq. (13) but N terms drop out of Eq. (12). Thus if one identify Eq. (12) and Eq. (13) following conditions must be hold simultaneously

$$c_{mn} = a_m b_n = 0 \; \text{and} \; a_m, b_n \neq 0. \qquad (14)$$

Of course zero divisors (an element which satisfy Eq. (14)) do not occur in Hilbert space.

Now we remind of some facts of algebra [2]. A ring is an additive abelian group with second binary operation denoted as multiplication. A ring is a field if it is commutative and its non-zero elements form a group under multiplication. Hence for each no-zero element $f \in F$ there is $f^{-1} \in F$. If M is an additive abelian group and R a ring than the statement that M is R-module means

$$\begin{aligned} &M \times R \to M \\ &(m, r) \to m \cdot r. \end{aligned} \qquad (15)$$

This mapping must satisfy some conditions which are not important for this moment. If F is a field then F-module is called vector space. One can now see that the multiplication operation between transition amplitudes $a_m b_n$ is not defined. This is the reason why Eq. (12) and Eq. (13) are not equal. Hence if there is a demand given by Eq. (14) then either $a, b \in R$ or statement (14) is meaningless. Let us try to define the mapping $R \times F \to R$. This mapping is not possible (in physical sense) because than $a \in F$ not R. The only possible structure is $R_1 \times R_2 \to R_1$. But then if $v \in R_2$ is a zero divisor then

$$\|v\| = 0. \qquad (16)$$

Recall that v represents a state of a physical system. However Eq. (16) represents non-physical state (the only state satisfying Eq. (16) is zero state). The conclusion is that ring can not represent a physical system and Eq. (12) and Eq. (13) can not be identified.

4. COHERENT SOURCE

If a light source emits a huge number of photons, the whole system of these particles can be written by using a Fock space. The advantage of this notation is that the number of particles need not be specified. In particular if n_μ is the number of photons in state $|\mu\rangle$ then the normalized basis states in Fock Space are [1]

$$|...n_\mu...\rangle = \prod_\mu (n_\mu!)^{-1/2} (a_\mu^+)^n |0\rangle, \qquad (17)$$

where $|0\rangle$ denotes the normalized state in which no particle are presented and a_μ^+ denotes the creation operator (when a_μ^+ is acted on $|0\rangle$ then in the outcome state there is one particle in the state $|\mu\rangle$).

We now assume laser radiation. Hence there is only one possible state in which photons can occurred. Hence one can write

$$|n\rangle = (n!)^{-1/2} (a^+)^n |0\rangle. \qquad (18)$$

So basis states are defined by Eq. (18). The next task is to find complex coefficients which completely specify a coherent state. These coefficients can be easily found because it is well know from classical physics that probabilities of finding exactly n photons in the coherent state satisfy Poisson distribution. Using this outcome of classical physics is not in conflict with concept of quantum mechanics. The reason is following. The result of a measurement of a quantum system must be expressed in the classical way. This claim is given by the fact that measuring devices are classical (macroscopic). Moreover the fact that photons in coherent state satisfy Poisson distribution was experimentally observed.

However from the mentioned condition of Poisson distribution one can deduce that

$$\langle n|\alpha\rangle\langle\alpha|n\rangle = \mathrm{e}^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}, \qquad (19)$$

where $|\alpha\rangle$ denotes the coherent state and $|\alpha|^2 = \langle\alpha|N\alpha\rangle = \langle n\rangle$ denotes the average number of photons in state $|\alpha\rangle$. From the superposition principle, Eq. (18) and Eq. (19) one can obtain a familiar formula

$$|\alpha\rangle = \mathrm{e}^{-(1/2)|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \qquad (20)$$

Eq. (20) is ambiguous. This is given by complex notation of Hilbert space.

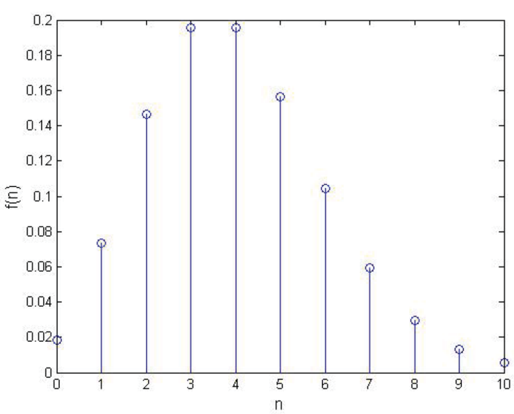


Fig. 1. Poisson distribution for $\langle n\rangle = 4$.

5. BACKGROUND RADIATION

In free space optics background radiation play an important role. Background radiation can be seen as a special type of light source. The number of irrelevant degrees of freedom depends on measuring device. If photodiode is used as a measuring apparatus then the measurement is performed in basis $|n\rangle$. Hence to completely specify background radiation source one must expressed the density matrix in the number states representation. Hence next task is to find the following matrix elements

$$\begin{array}{ccccc} & \boldsymbol{\rho}|0\rangle & \boldsymbol{\rho}|1\rangle & .. & \boldsymbol{\rho}|n\rangle \\ |0\rangle & ? & 0 & 0 & 0 \\ |1\rangle & 0 & ? & 0 & 0 \\ : & 0 & 0 & ? & 0 \\ |n\rangle & 0 & 0 & 0 & ? \end{array}. \qquad (21)$$

The remaining matrix elements can by obtained by using the following consideration. The statement that photodiode registers exactly n photons means that events $n = 0, 1, 2, ..., n-1$ do not occur. In the other words diagonal elements of $\boldsymbol{\rho}$ are equivalent to the question: how many independent attempts is needed to do until an event n occurs. A sequence of these independent attempts is called Bernoulli sequence.

Hence one can guess that the finding probability distribution is Negative binomial distribution given by following formula

$$f(x_i) = \binom{x_i + m - 1}{m-1}\cdot p^m(1-p)^{x_i}, \qquad (22)$$
$$x_i = 0, 1, 2, ...$$
$$0 < p < 1 \text{ and } m = 1, 2, ...$$

where p denotes the probability of occurring event n (exactly n photons are detected) and m denotes the number of occurring events n. Hence in our task $m = 1$. If the condition $m = 1$ is hold then one obtains from Eq. (22)

$$f(x_i) = p(1-p)^{x_i}. \qquad (23)$$

Probability distribution (23) is also called Geometric distribution. Geometric distribution is shown in Fig. 1.

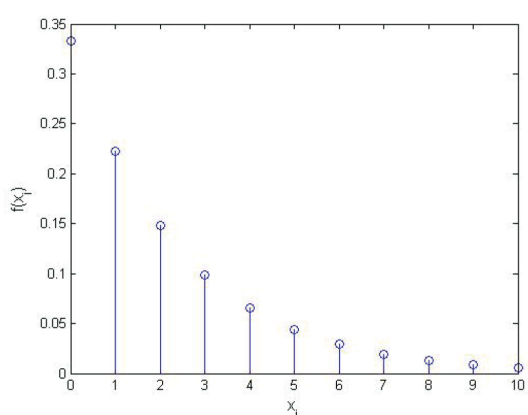


Fig. 1. Geometric distribution for $p = 1/3$

The diagonal elements therefore satisfy Eq. (23) where $x_i \equiv n$.

Up to now background radiation was not quantitatively characterized. To solve this problem we calculate the average value and the variation of Geometric distribution

$$E(X) = \frac{1-p}{p},\ D(X) = \frac{1-p}{p^2}. \qquad (24)$$

Now one can deduce according to the analogy with Poisson distribution that the average value in Eq. (24) characterizes background radiation source. So one can equate $E(X) = \langle n_{BR} \rangle$ where $\langle n_{BR} \rangle$ denotes the average number of photons which quantitatively characterized background radiation source. With using Eq. (24) one can calculate

$$p = \frac{1}{1+\langle n_{BR} \rangle}. \qquad (25)$$

The boundary value for $\langle n_{BR} \rangle = 0$ is $p = 1$. This is not in contradiction with conditions stated in Eq. (22) because $m = 1$.

6. SUMARY

In the article there was given a review of description of basic states using in quantum communication.

Concretely there was stated the general wave function of a photon and shown how to determine the density matrix for a single photon source. If the wave function can not be factorized into terms which correspond to the particular systems then the concept of entanglement appears. An entanglement can not be seen as that is something strange or that is an exception because an entanglement is correctly expressed using superposition principle. If one tries to identify the tensor product and superposition principle he runs into problem. This problem can not be resolved by finding a new mathematical structure in which quantum theory acts. The reason is that this new structure does not satisfy physical properties.

Next in the paper there was shown the familiar method how to describe a huge number of photons. This method was applied to coherent states.

Finally the main goal of the article was the determination of density matrix of background radiation source. It was shown that matrix elements satisfy Geometric distribution in the number state representation.

7. ACKNOWLEDGEMENT

Research described in the paper was financially supported by the research programs GAČR č. 102/05/0732, 102/05/0571 and 102/03/H109.

8. REFERENCES

1. Peres A., *Quantum Theory: Concepts and Methods*, New York, Boston, Dordrecht, London, Moscow: Kluwer Academic Publishers, 2002, ISBN 0-306-47120-5.
2. Connell E.H., *Elements of Abstract and Linear Algebra*, eBook, 2004.
3. Zapletal J., *Základy počtu pravděpodobnosti a matematické statistiky*, Brno, 1995, ISBN 80-214-0711-5.