Weakly and strongly nonlinear waves in negative phase metamaterials

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ABSTRACT

A fundamental approach to a slowly varying amplitude formulation for nonlinear waves in metamaterials will be established. The weakly nonlinear slowly varying amplitude approach will be critically examined and some misunderstandings in the literature will be fully addressed. The extent to which negative phase behaviour has a fundamental influence upon soliton behaviour will be exposed. The method will deploy nonlinear diffraction and a special kind of diffraction-management. This is additional to a detailed modulation instability analysis. The examples given involve waveguide coupling and a nonlinear interferometer. In addition, a strongly nonlinear approach will be taken that seeks exact solutions to the nonlinear equations for a metamaterial. A boundary field amplitude approach will be developed that leads to useful eigenvalue equations that expose, in a very clear manner, the possibility that new kinds of waves can be generated.

Keywords: Metamaterial, Negative Refraction, Soliton, Nonlinear Schrödinger Equation, Nonlinearity, Modulation.

1. INTRODUCTION

Since the first observation[1] of a soliton by John Scott Russell in 1834, the potential for its application has been slowly realised with a considerable impetus from the optical fibre domain[2]. The kind of soliton that is of interest in the optical frequency range is analogous to solitons that can be generated on deep water and these contrast with the KdV solitons observed by Russell on shallow water[1]. For optical systems, soliton research often divides up into an emphasis upon dispersion or diffraction. If both are considered simultaneously then the emphasis would be upon so called optical bullets, or optical machine guns[3], but if diffraction alone is taken as the competitor to nonlinear self-focussing, then it is an area of research that comes under the heading called spatial solitons[4,5]. It is the latter that will be discussed in detail in this paper with a special emphasis on how to exploit negative phase metamaterials[6-8] to achieve certain types of soliton behaviour.

Even a rapid examination of the literature concerning solitons shows that they are members of a large family[1]. It is therefore interesting to select the appropriate family member for the system under investigation. KdV solitons posses a speed that is proportional to their amplitude, but these are not the same kind of solitons that occur on deep water[1]. Waves on deep water are the hydrodynamic analogues. In fact so-called Stokes waves are the surface waves on deep water, where the latter comment simply means that the depth is much larger that the wavelength. Such waves are unstable with respect to perturbations, or more precisely, modulations, as Lighthill was able to show in 1967. Just as Russell was not believed in his time, the result published by Lighthill was also met with widespread incredulity[9]. Of course Lighthill was correct and it is now recognised that a modulation instability regime is a necessary condition for the existence of solitons[2]. For an optical beam, it is the issue of self-focussing competing with diffraction that is resolved by the formation of what can be turned as a spatial soliton[4,5]. Just as in the situation that arises when considering wave packets an investigation of modulation instability will lead to the identification of soliton formation. The outcome of such an instability analysis is that the wave number associated with the propagation is pure imaginary over a substantial spatial frequency region and that this leads to gain.

A major property of metamaterials[10] is their ability to support backward waves[6-8,11,12] and this is achieved by the creation of artificial materials that posses both dielectric and magnetic properties expressed through the creation of frequency dependent relative permittivity and permeability. For such materials the phase accumulation for forward waves can be the exact opposite of the phase accumulation for backward waves. Clearly this would require the creation of

homogeneous, isotropic metamaterials and this possibility will be assumed below. It is also a current preoccupation to be very concerned about losses. For the majority of the work set out below, it will be assumed that losses have been taken care of by the creation of a suitable active metamaterial, but it should be pointed out that losses can be safely taken into the models provided that dissipative solitons are the accepted outcomes[13].

The discussion presented here will begin with a derivation of the nonlinear Schrödinger equation that describes the slowly varying envelope of an optical beam that is propagating through an isotropic, loss-free, double-negative metamaterial. This will then be used to engage in a form of diffraction-management[14,15] and this will lead not only to a fascinating modulation instability analysis, but also to some other coupled state examples. The concluding part of the paper deals with what will be termed here "strongly" nonlinear waves. It will be demonstrated that the latter do not invoke a slowly varying amplitude and that novel surface waves are an expected outcome. Finally some comments will be made about optical bullets and a progression towards the inclusion of damping and the possibility of magnetooptic tuning.

2. SPATIAL NONLINEAR SCHRÖDINGER EQUATION

It is appropriate at this stage to use the Fourier transforms with respect to time, so that the angular frequency, ω, is introduced, and the field vectors are transformed into the frequency domain. The electric field, magnetic flux density, the magnetic field and the displacement vectors governed by Maxwell's equations are defined here as
$E \equiv E(\omega), B \equiv B(\omega), H \equiv H(\omega), D \equiv D(\omega)$. Now that the definitions are clarified, it will be assumed that from now on it is Fourier component that are being used. Given this notation, the constitutive material relationships can be written in terms of a nonlinear polarization P_{NL} associated with the dielectric behaviour and a nonlinear magnetization M_{NL} associated with the permeability. For the moment the nonlinear magnetization will not be used, choosing instead to invest the dielectric with nonlinear properties. Given this assumption,

$$D = \varepsilon \square E + P_{NL} \qquad\qquad B = \mu \square H \qquad\qquad (1.1)$$

$$\mu = \mu_0 \mu(\omega) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad\qquad \varepsilon = \varepsilon_0 \varepsilon(\omega) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad\qquad (1.2)$$

where the relative permittivity $\varepsilon(\omega)$ and the relative permeability $\mu(\omega)$ are assumed to be frequency dependent and describe an isotropic, homogeneous material. It is not the intension at this stage to admit that these properties are complex because of the presence of damping.

It is common practice, as this stage, to assume that $\nabla \square E = 0$, but this far-reaching assumption needs to be challenged because it is not a fundamental member of the complete set of Maxwell's equations, whereas, indeed, $\nabla \square D = 0$ is such a member. If the usual assumption about $\nabla \square E$ is not made, then the wave equation, in this case, is[16]

$$\nabla \left(-\frac{1}{\varepsilon_0 \varepsilon(\omega)} \nabla \square P_{NL} \right) - \nabla^2 E = \frac{\omega_0^2}{c^2} \mu(\omega) \varepsilon(\omega) E - \frac{\omega_0^2}{c^2} \frac{\mu(\omega)}{\varepsilon_0} P_{NL} \qquad\qquad (1.3)$$

This equation can be developed to form a description of spatial solitons of the type sketched in Fig. 1. Here a beam propagates down the z axis, diffracts along x and is guided with respect to the y direction.

SPATIAL SOLITONS

Z : Direction of propagation
Y : Freezing due to guiding structure
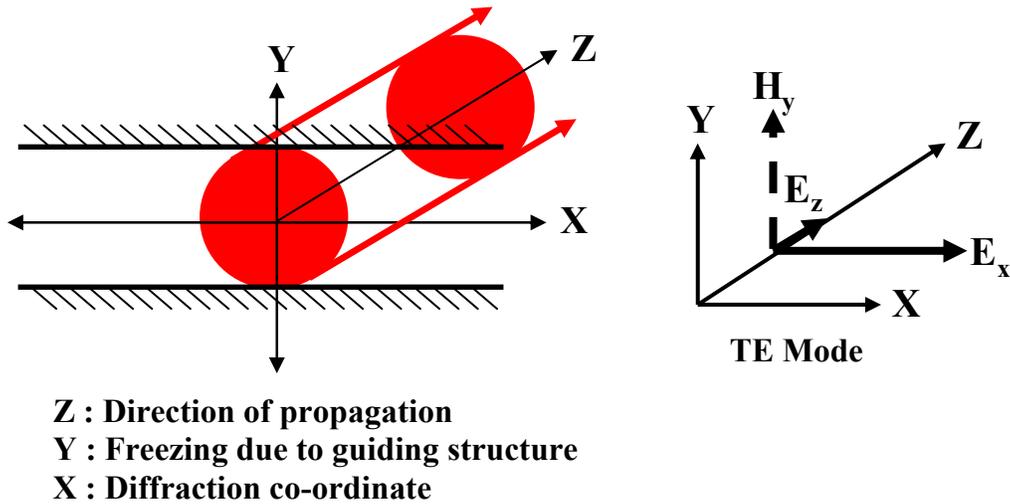**X : Diffraction co-ordinate**

Fig. 1. A sketch of how a spatial soliton can be formed within a planar waveguide. The propagating mode is an envelope that is quasi-TE

Even though the nonlinearity in the permittivity is assumed to be dominant, this is not a major restriction, given that typically a nonlinear permeability contributes a term that is the order of $|\boldsymbol{H}|^2$ and that $\boldsymbol{H}$ is simply related to $\boldsymbol{E}$. This is an important point because if there is nonlinearity only in the permittivity, it is obviously possible to obtain an envelope equation purely in terms of a single field component, hence it may be thought that when both ε and μ are nonlinear, this will inevitably lead to coupled envelope equations. In fact this has already been stated in the literature[17] and is *incorrect reasoning*. Any contribution from the μ can be rolled into the contribution from the permittivity, resulting in a modified effective nonlinear dielectric susceptibility. In general, for the TE wave sketched in Fig. 1. The components of the nonlinear polarization, assuming a homogeneous, isotropic, dielectric are[4,5]

$$P_{NLx} = \varepsilon_0 \chi_{xxxx} \left(\frac{3}{4} |E_x|^2 E_x + \frac{1}{2} |E_z|^2 E_x + \frac{1}{4} E_z^2 E_x^* \right) \tag{1.4}$$

$$P_{NLz} = \varepsilon_0 \chi_{xxxx} \left(\frac{3}{4} |E_z|^2 E_z + \frac{1}{2} |E_x|^2 E_z + \frac{1}{4} E_x^2 E_z^* \right) \tag{1.5}$$

where χ_{xxxx} is the relevant tensor component of the third order nonlinear susceptibility

However, if there is only weak guiding then it has been shown[18] that the longitudinal component is negligible, i.e. $|E_z| << |E_x|$. Given this fact, it is safe to neglect terms involving E_z in equations (1.4) and (1.5). In practice, therefore, only P_{NLx} survives and it is convenient to drop the subscript x, so equation (1.4) becomes

$$P_{NL} = \frac{3}{4} \varepsilon_0 \chi^{(3)} |E|^2 E \tag{1.6}$$

where, for convenience the nonlinear susceptibility is written as $\chi^{(3)} = \chi_{xxxx}$

Only the x-component of the polarization survives and a spatial soliton can propagate as a beam launched at a particular frequency ω_0 that is associated with a wave number k_0, where

$$k_0^2 = \frac{\omega_0^2}{c^2}\left(\mu(\omega_0)\varepsilon(\omega_0)\right) \tag{1.7}$$

The electric field component can be decomposed into a slowly varying envelope $A(x,t)$ and a rapidly varying (fast) carrier wave $e^{i(k_0 z - \omega_0 t)}$. The final envelope equation used here is

$$2ik_0\frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial x^2} + \frac{3\omega_0^2}{4c^2}\mu(\omega_0)\chi^{(3)}|A|^2 A + \frac{3\chi^{(3)}}{4\varepsilon(\omega_0)}\frac{\partial^2}{\partial x^2}\left(|A|^2 A\right) = 0 \tag{1.8}$$

Earlier on it was emphasized that the component of field along the propagation direction is negligible, so that non-paraxiality has been ignored. This means that the opportunity to include extra terms revolves around retaining the final term in (1.8) and the possibility of higher-order diffraction. Later on the main diffraction term will be modified, or managed to such an extent that the final term in (1.8) will come into play as diffraction, and, in fact, it is called the *nonlinear diffraction* term. It has been shown in the past[16] that this term is the dominant correction, even though it is the order of a possible quintic correction.

The first point to make is that in a negative phase medium, k_0 is negative, so that $k_0 = -|k_0|$ and $\mu(\omega_0) = -|\mu(\omega_0)|$, $\varepsilon(\omega_0) = -|\varepsilon(\omega_0)|$. Hence for a metamaterial eqn (1.8) becomes

$$i\frac{\partial A}{\partial z} - \frac{1}{2|k_0|}\frac{\partial^2 A}{\partial x^2} + \frac{3\omega_0^2}{8c^2}\frac{|\mu(\omega_0)|\chi^{(3)}}{|k_0|}|A|^2 A + \frac{3\chi^{(3)}}{8|\varepsilon(\omega_0)||k_0|}\frac{\partial^2}{\partial x^2}\left(|A|^2 A\right) = 0 \tag{1.9}$$

Provided that higher-order terms and non-paraxiality are not taken into account, eqn (1.9) has the same structure as a form produced earlier[19], but it is emphasized that, once again, that coupled equations are not produced. This is in contradiction to a recent publication[17].

3.   NORMALISATION AND DIFFRACTION MANAGEMENT

At this stage, the sign in front of the main diffraction term is negative so that a self-defocusing nonlinearity is required. The sign in front of the nonlinear diffraction term would then also be negative. If very narrow solitons are required in the form of optical needles it would be very useful to manipulate the main diffraction term. This can be done by proposing that solitons propagate through an alternating periodic medium, during which they experience a positive wave number followed by a negative wave number as they cross the various regions. It is easy to imagine how such a structure could be created lithographically, even including a suitable deposition that accounts for any possible unwanted reflections due to impedance mismatching. The kind of structure envisaged is sketched in Fig. 2.
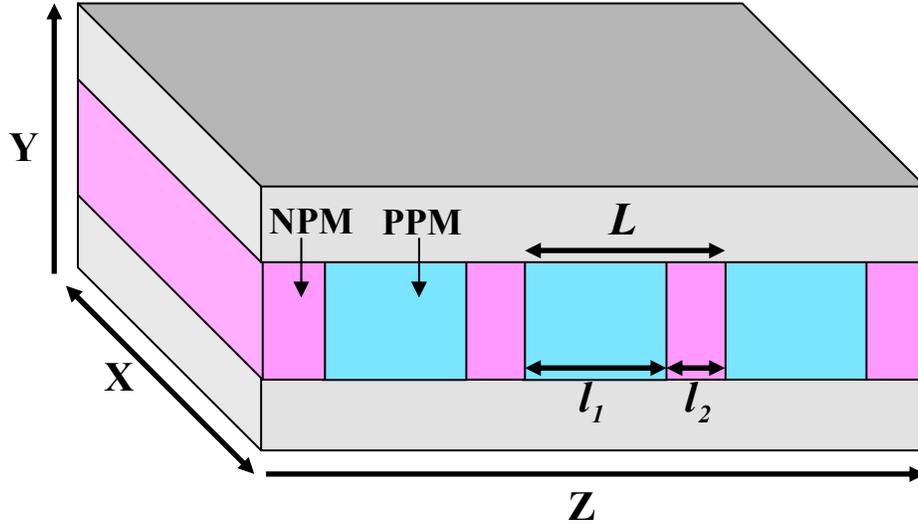
Fig. 2. A diffraction-managed waveguide. PPM is a positive phase medium, and NPM is a negative phase medium.

The idea of balancing a PPM and an NPM has previously been discussed in the context of cavities[15]. Clearly, L is much bigger than a wavelength and the latter is larger that the inter-metaparticle spacing. It is assumed here that impedance matching is in place. After averaging over the PPM/NPM structure displayed in Fig. 2. the main diffraction term, assuming that the nonlinearity is confined to the PPM, the following parameter appears.

$$D = l_1 - \frac{k_{01}}{|k_{02}|} l_2 \tag{2.1}$$

where k_{01} is the wave number of the fast variation in the PPM and k_{02} is the wave number of the fast variation in the NPM. After scaling x and z to be $x = wX$ and $z = k_{01}w^2 Z$, where w is the input beam width, and adopting the following transformation for A

$$\psi^2 = \frac{3w^2}{8} \frac{\omega^2}{c^2} \chi^{(3)} \mu_1 l_1 A^2 \tag{2.2}$$

the final envelope equation, to be used here, takes the form

$$i \frac{\partial \psi}{\partial Z} + \frac{D}{2} \frac{\partial^2 \psi}{\partial X^2} + |\psi|^2 \psi + \kappa \frac{\partial^2}{\partial X^2} \left(|\psi|^2 \psi \right) = 0 \tag{2.3}$$

where the nonlinear diffraction enters into the description through the parameter

$$\kappa = \frac{1}{w^2 k_{01}^2} \tag{2.4}$$

4.   MODULATION INSTABILITY ANALYSIS

The foregoing analysis has produced a modified form of the nonlinear Schrödinger equation. Even though it may appear that the metamaterial properties have been buried within the equation, it is apparent that the diffraction can be nicely managed using a combination of positive and negative phase media. Furthermore, the very interesting nonlinear

diffraction property has been included. The latter is a very important step because this nonlinearly induced-diffraction assumes a dominant role as the solitons become very narrow and needle-like. It is so dominant, in fact, that it supersedes the need to account for non-paraxiality that is sometimes invoked to prevent catastrophic collapse of soliton beams[16]. In this section, the impact of diffraction-management and nonlinear diffraction on modulation instability is discussed. As is well known, modulation instability is a precursor to soliton formation[2], and forms a valuable stepping stone to the investigation of soliton behavior that can include special interactions of the kind associated with potential optical chips. Modulation instability analysis is based upon investigating what happens to a steady state solution when a perturbation is added. It has already been investigated extensively for metamaterials with the main outcome being a switch in sign for the self-steepening term[20-22]. Naturally, modulation instability analysis could apply to solitons themselves, but in this section it is the traditional plane wave steady state solution of the nonlinear Schrödinger equation that is analyzed. Such a steady state solution can be written as

$$\psi = \sqrt{P_0}\, e^{iP_0 Z} \tag{3.1}$$

where P_0 is the normalized power. A small perturbation can be added to this in the form

$$\psi = \left(\sqrt{P_0} + a\right) e^{iP_0 Z} \tag{3.2}$$

Taking into account only first-order terms in a, leads to

$$i\frac{\partial a}{\partial Z} + \frac{D}{2}\frac{\partial^2 a}{\partial X^2} + P_0\left(a + a*\right) + \kappa P_0 \frac{\partial^2}{\partial X^2}\left(a + a*\right) = 0 \tag{3.3}$$

The following solution is now adopted

$$a = a_1 e^{i(KZ - \Omega X)} + a_2 e^{-i(KZ - \Omega X)} \tag{3.4}$$

After the substitution of eqn (3.4) into eqn (3.3), each term of the differential equation produces, respectively,

$$i\frac{\partial a}{\partial Z} = -K a_1 e^{i(KZ - \Omega X)} + K a_2 e^{-i(KZ - \Omega X)} \tag{3.5}$$

$$\frac{D}{2}\frac{\partial^2 a}{\partial X^2} = \frac{D}{2}\left(-\Omega^2 a_1 e^{i(KZ - \Omega X)} - \Omega^2 a_2 e^{-i(KZ - \Omega X)}\right) \tag{3.6}$$

$$P_0\left(a + a*\right) = P_0\left(a_1 e^{i(KZ - \Omega X)} + a_2 e^{-i(KZ - \Omega X)} + a_1 e^{-i(KZ - \Omega X)} + a_2 e^{i(KZ - \Omega X)}\right) \tag{3.7}$$

$$\kappa P_0 \frac{\partial^2}{\partial X^2}\left(a + a*\right) = \kappa P_0\left(-\Omega^2 a_1 e^{i(KZ - \Omega X)} - \Omega^2 a_2 e^{-i(KZ - \Omega X)} - \Omega^2 a_1 e^{-i(KZ - \Omega X)} - \Omega^2 a_2 e^{i(KZ - \Omega X)}\right)$$
$$\tag{3.8}$$

Collecting all the co-efficients of the different exponentials yields the following

$e^{i(KZ - \Omega X)}$:
$$-K a_1 - \frac{D}{2}\Omega^2 a_1 + P_0 a_1 + P_0 a_2 - \kappa P_0 \Omega^2 a_1 - \kappa P_0 \Omega^2 a_2 = 0 \tag{3.9}$$

$e^{-i(KZ-\Omega X)}$:

$$Ka_2 - \frac{D}{2}\Omega^2 a_2 + P_0 a_2 + P_0 a_1 - \kappa P_0 \Omega^2 a_2 - \kappa P_0 \Omega^2 a_1 = 0 \qquad (3.10)$$

In matrix form, the latter equations become

$$\begin{pmatrix} -K - \dfrac{D}{2}\Omega^2 + P_0 - \kappa P_0 \Omega^2 & P_0 - \kappa P_0 \Omega^2 \\ P_0 - \kappa P_0 \Omega^2 & K - \dfrac{D}{2}\Omega^2 + P_0 - \kappa P_0 \Omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0 \qquad (3.11)$$

and the determinant of the matrix in (3.11) evaluates to

$$K^2 = \frac{D^2}{4}\Omega^4 - DP_0\Omega^2 + D\kappa P_0 \Omega^4 \qquad (3.12)$$

The solution of this is simply

$$K = \pm\frac{D|\Omega|}{2}\sqrt{\frac{4\kappa P_0 \Omega^2}{D} - \frac{4P_0}{D} + D\Omega^2} \qquad (3.13)$$

and this now permits a straight forward interpretation. The early analysis has proceeded on the basis that a perturbation will vary as $\exp(\pm iKZ)$ so, if there is the possibility that K can become complex, or pure imaginary, gain will be implied.

In fact, $g(\Omega)$, the gain of the system, in this case, is twice the imaginary part of K, i.e.

$$g(\Omega) = 2\,\mathrm{Im}(K) = \sqrt{D}\,|\Omega|\sqrt{4P_0 - 4\kappa P_0 \Omega^2 - D\Omega^2} \qquad (3.14)$$

For $\kappa = 0$ the nonlinear diffraction is absent, and this is the full diffraction case $(D=1)$, and the result reduces to the well-known form[2], with $\Omega_{max} = \sqrt{2P_0}$ and $g_{max} = 2P_0$. If $D \neq 1$ and $\kappa \neq 0$ then

$$\Omega_{max}^2 = (4P_0)/(2D + 8P_0\kappa) \qquad (3.15)$$

and

$$g_{max} = (2P_0\sqrt{D})/\sqrt{4\kappa P_0 + D} \qquad (3.16)$$

The important point to remember is that whenever a beam exists, its Fourier transform yields the transverse spatial frequency range, Ω. A narrow beam in the x space direction implies that it is a rather broad beam in Ω-space. Hence it can be expected that the gain region will extend over a substantial range of Ω. Normally, a beam of modest width will have an Ω spectrum such that its half width does not extend very far down the Ω-axis. In all modulation instability analyses the gain goes to a maximum and the location of this maximum is related to the half-width that exists in Ω-

space. At $\Omega = 0$, the transverse wave number corresponds to an infinite wavelength and, hence, it is like a plane wave, so no gain exists. As Ω increases the effective transverse wavelength decreases so there will come a point at which diffraction and self-focusing can not balance, so the gain is brought abruptly to an end. In the case of diffraction-management, very narrow, optical needle-like beams become possible so the range of Ω in the Fourier domain becomes very large. In fact, really very small effective transverse wavelengths are operative in the beam. This means that the position on the Ω-axis, where the gain cuts off, proceeds further, and further, down the scale. These features are precisely shown in Fig. 3.
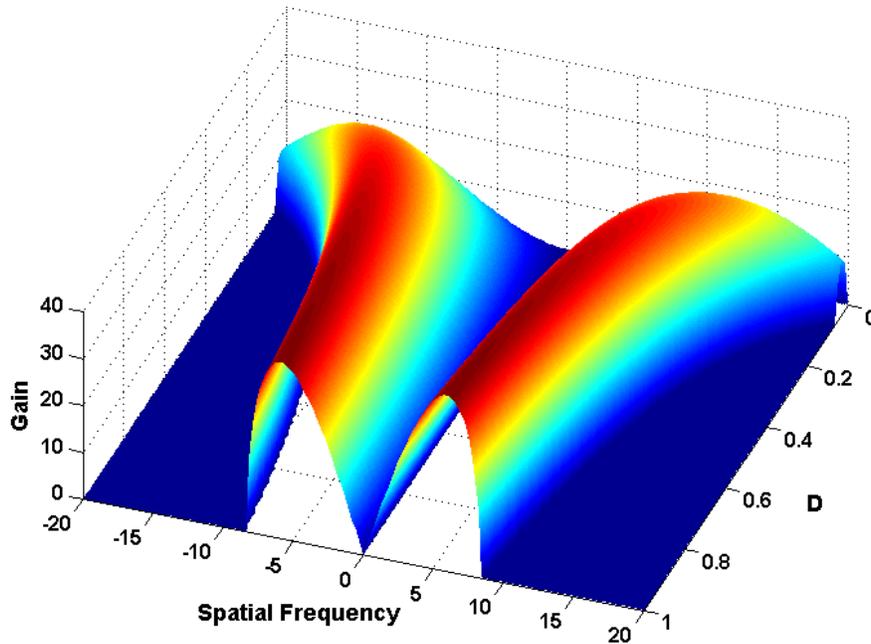

Fig. 3. Regions of gain for a range of diffraction management parameters against transverse wave number. $\kappa = 0.0018$ and $P_0 = 20$.

5.   SOLITON FORMATION AND COUPLING

The preceding sections lead to a theory of spatial soliton formation in a special metamaterial that is being combined with a positive phase medium i.e. a normal positive index material. This is referred to as diffraction-management and, as can be seen, a useful envelope equation can be developed. Furthermore, a modulation instability analysis leads to interesting conclusions. In this section, the opportunity is taken to consider, first of all, how a soliton is actually formed and to examine briefly how sensitive it is to the initial condition concerning the diffraction-management. Secondly, the literature shows that the manner in which solitons interact is always fascinating. Rather than deal directly with diffraction-managed soliton interactions in the bulk, this section takes a more interesting direction and investigates how diffraction management affects a nonlinear coupler[23]. The latter have many potential applications for all-optical operations[24,25] on the computing chips of the future, so it is useful to set up a relatively straight forward problem involving bright spatial soliton dynamics in a symmetric planar waveguide structure[4].
First of all, Fig. 4. shows generic soliton formation and, in particular, demonstrates a role for the nonlinear diffraction introduced earlier on.
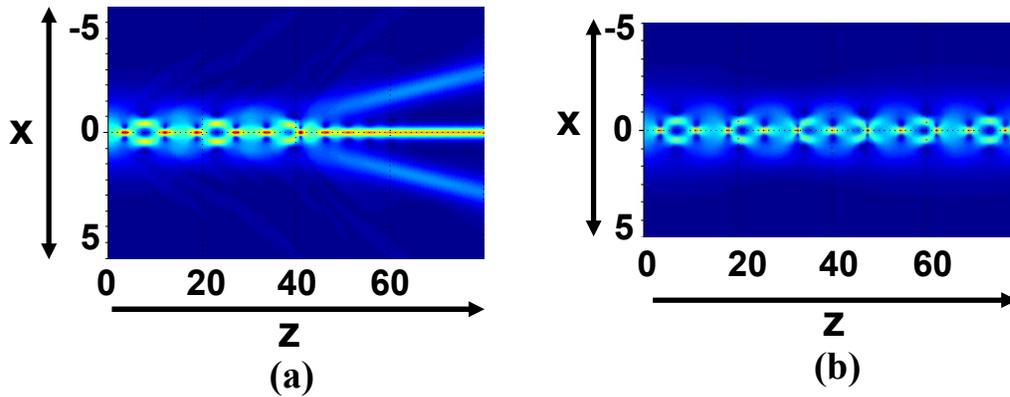
Fig. 4. Generic pictures of soliton formation for D =10 %. Z is the direction of propagation and is measured in Rayleigh lengths, and X is the diffraction direction, and is measured in beam widths. (a) In the absence of nonlinear diffraction an initial $\psi(x,0) = sech(x)$ input yields a low order breather. (b) Addition of nonlinear diffraction causes breather to break up. D = 10% including nonlinear diffraction with $\kappa = 0.0018$.

Fig. 4. demonstrates that, even if an unstable high-order soliton is created, the nonlinear diffraction introduces stability, at a point dependent upon the value of κ.
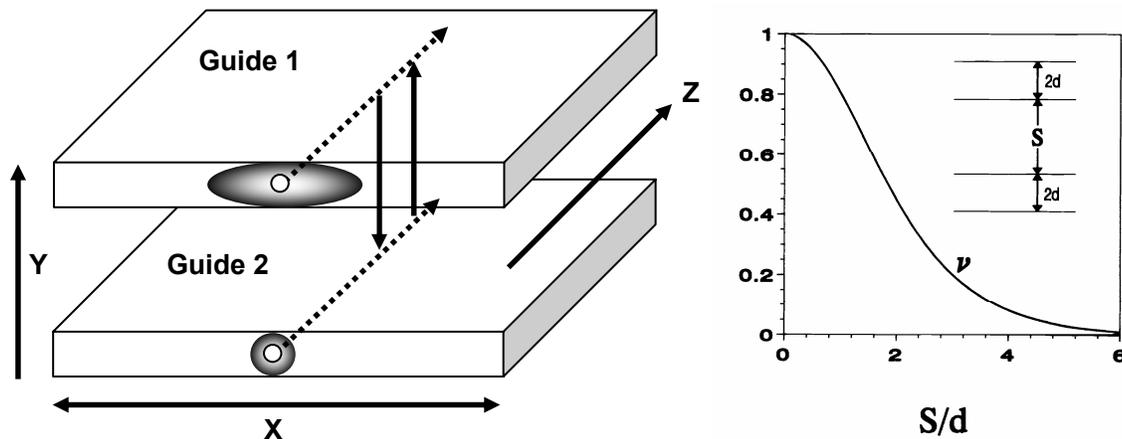The next set of results for this section concerns the following coupler arrangement.

Fig. 5. Sketch of a planar waveguide structure. Guide 1 is made of a normal positive phase medium. Guide 2 is diffraction managed. The graph showing V sketches the extent of linear coupling as a function of guide separation.

The interest here lies in the extent to which beams can communicate between the two guides. The simulations shown here refer to a particular form of coupling in which the envelope equation in guide 1 is coupled to guide 2 through a quantity $v\psi_2$ and guide 2 is coupled to guide 1 through a quantity $v\psi_1$. Hence, the appropriate coupled equations are

$$i\frac{\partial \psi_1}{\partial Z} + \frac{D}{2}\frac{\partial^2 \psi_1}{\partial X^2} + |\psi_1|^2 \psi_1 + \kappa \frac{\partial^2}{\partial X^2}\left(|\psi_1|^2 \psi_1\right) + v\psi_2 = 0 \qquad (4.1)$$

$$i\frac{\partial \psi_2}{\partial Z} + \frac{D}{2}\frac{\partial^2 \psi_2}{\partial X^2} + |\psi_2|^2 \psi_2 + \kappa \frac{\partial^2}{\partial X^2}\left(|\psi_2|^2 \psi_2\right) + v\psi_1 = 0 \qquad (4.2)$$

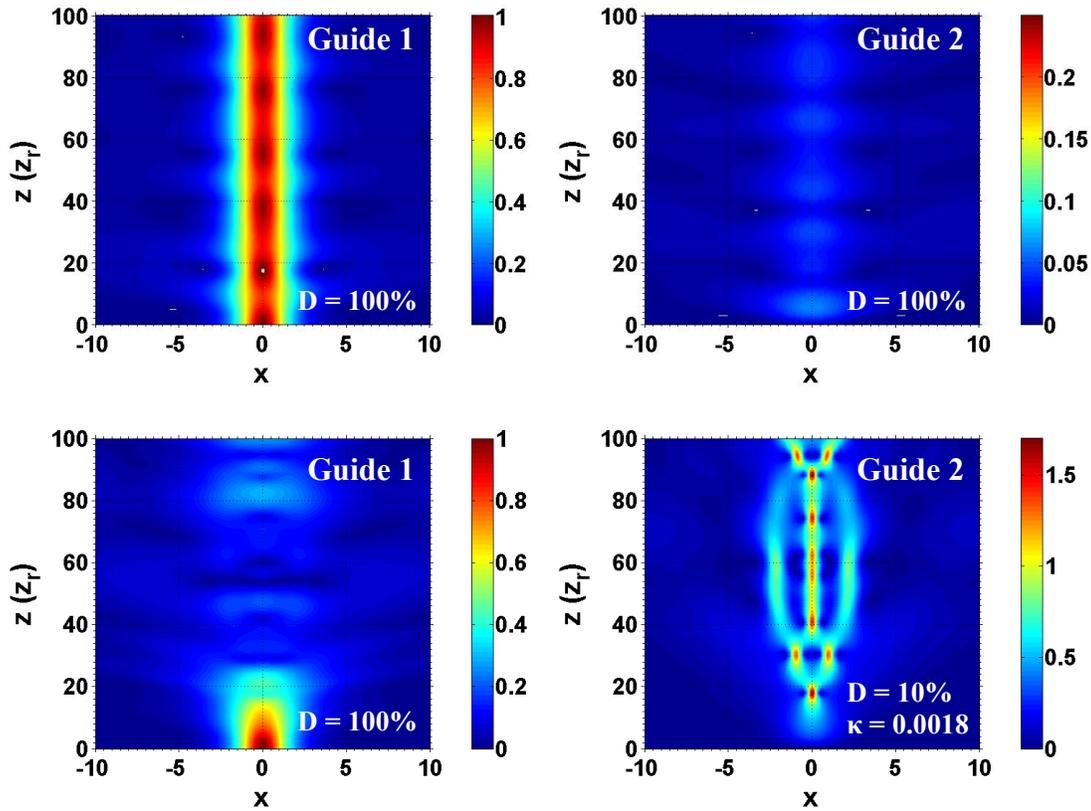These equations have been solved numerically and some typical examples are displayed below.

Fig. 6. The coupling between a pair of waveguides is shown for two cases. For the upper set D = 100% in both waveguides, and the small linear co-efficient $v = 0.1$ prevents a transfer of energy. In the lower set, both diffraction management and nonlinear diffraction facilitates a strong energy transfer. The initial input in both cases is $\psi(x,0) = sech(x)$ to guide 1.

For the coupler shown, only linear coupling is considered. As pointed out in the literature[23,] this is a particular type of nonlinear directional coupler. As can be seen in the upper simulations, a small amount of energy does transfer to guide 2, but because reducing D leads to strong focusing and very narrow solitons, the lower simulations show that the second guide not only receives an energy transfer, but more or less captures it completely.

6.   STRONGLY NONLINEAR WAVES

If the nonlinearity is strong, the shape of the modal fields of nonlinear guided waves changes significantly with power, as demonstrated in the widely cited papers by Boardman and Egan. Fir this regime, the first-order coupled mode approach is quite inadequate. This is the situation that will be considered in this section, and will require exact solutions of the nonlinear equations. The dielectric tensor will be assumed to be centrosymmetric and the nonlinearity to be unsaturated, gaining its nonlinearity from a third-order polarization. As indicated earlier, it is straightforward to include this type of nonlinearity into the permeability as well with a suitable modification of the effective nonlinear coefficient. Hence a quadratic dependence of the nonlinear refractive index is adopted and waves that are at the fundamental frequency ω and 3ω can be created. The generation of the third-harmonic will be assumed to be a, poorly phase-matched, small effect. The possibility that the nonlinearity will saturate and the role of any absorption process will be addressed elsewhere. It is the Kerr regime that is being adopted, therefore, and it will be examined through an investigation of TE waves in a nonlinear slab guide that has linear bounding material. Both the relative permittivity and the relative permeability are less that zero and, hence, are in the negative phase frequency range of operation. The damping is not considered here on the grounds that this is assumed to be an active medium, for which the loss is eliminated, or minimized to have a very small influence on the final outcomes.
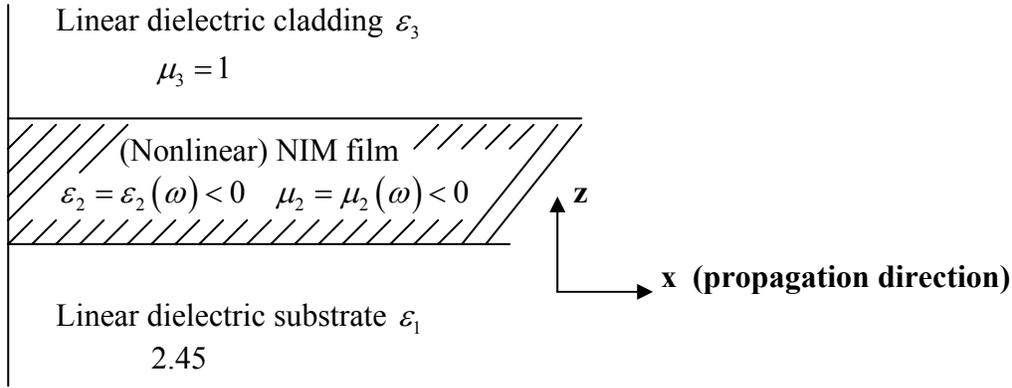
Fig. 7. A nonlinear slab of negative index material [NIM]

For TE waves propagating along the x-axis, with a wave number k_x, the equation for the exact amplitude is, expressed as a first integral,

$$\left(\frac{\partial A_2}{\partial z}\right)^2 - \left(k_x^2 - \frac{\omega^2}{c^2}\varepsilon_2(\omega)\mu_2(\omega)\right)A_2^2 + \frac{\omega^2}{c^2}\mu_2(\omega)\frac{\alpha}{2}A_2^4 = c_2 \qquad (5.1)$$

where c_2 is the integration constant, the nonlinear relative permittivity is $\varepsilon_{NL} = \varepsilon_2(\omega) + \alpha|A_2|^2$, and the linear relative permittivity and permeability are, respectively,

$$\varepsilon_2(\omega) = \varepsilon_B - \frac{\omega_p^2}{\omega^2}, \qquad\qquad \mu_2(\omega) = 1 - \frac{F\omega^2}{\omega^2 - \omega_0^2} \qquad (5.2)$$

As stated earlier, for the time being it is the nonlinearity associated with the relative permittivity that is accounted for but it would not be difficult to associate a Kerr-like nonlinearity with e permeability as well without making a substantial difference to the final outcomes. Note that ε_2 contains the background dielectric constant and that this is modified by the plasma frequency term from embedded wires, possibly, to give rise to a negative permittivity. The relative permeability is modeled by what might be termed the 'F model' in which $F = 0.56$ and $\omega_0 = 0.4\omega_p$. for the calculations reported here.

In the substrate and cladding, the first integrals are

$$\left(\frac{\partial A_{1,3}}{\partial z}\right)^2 - \left(k_x^2 - \frac{\omega^2}{c^2}\varepsilon_{1,3}(\omega)\right)A_{1,3}^2 = 0 \qquad (5.3)$$

for which the integration constants are zero because both the substrate and the cladding are of infinite extent. After adopting the following transformations to dimensionless quantities

$$E_i = \sqrt{\frac{\alpha}{2}}A_i; \quad \varpi^2 = \frac{\omega^2}{\omega_p^2}; \quad C_2 = \frac{c^2}{\omega_p^2}\frac{\alpha}{2}c_2; \quad Z = \frac{\omega_p^2}{c^2}z; \quad \kappa_2^2 = \frac{c^2k_x^2}{\omega_p^2} - \frac{\omega^2}{\omega_p^2}\mu_2\varepsilon_2 = \kappa_x^2 - \varpi^2\mu_2\varepsilon_2 \quad (5.4)$$

and the boundary conditions for TE waves, .

$$E_1\big|_{x=0} = E_2\big|_{x=0} = E_0; \quad E_3\big|_{x=d} = E_2\big|_{x=d} = E_b; \qquad \frac{\partial E_1}{\partial Z}\bigg|_{x=0} = \frac{1}{\mu_2}\frac{\partial E_2}{\partial Z}\bigg|_{x=0}; \quad \frac{\partial E_3}{\partial Z}\bigg|_{x=d} = \frac{1}{\mu_2}\frac{\partial E_2}{\partial Z}\bigg|_{x=d} \tag{5.5}$$

the following boundary field relationship emerges

$$\left[E_b^2 - \frac{\kappa_3^2\mu_2^2 - \kappa_2^2}{2\varpi^2|\mu_2|}\right]^2 - \left[E_0^2 - \frac{\kappa_1^2\mu_2^2 - \kappa_2^2}{2\varpi^2|\mu_2|}\right]^2 = \left[\frac{\kappa_3^2\mu_2^2 - \kappa_2^2}{2\varpi^2|\mu_2|}\right]^2 - \left[\frac{\kappa_1^2\mu_2^2 - \kappa_2^2}{2\varpi^2|\mu_2|}\right]^2 \tag{5.6}$$

which is the equation of a hyperbola and, although more complicated, is an example of the conic sections used in an earlier paper, where the equivalent solution can be found by putting $\mu_2 = 1$ in (5.6) to give

$$\left[E_0^2 + \frac{\varepsilon_1 - \varepsilon_2}{2}\right]^2 - \left[E_b^2 + \frac{\varepsilon_3 - \varepsilon_2}{2}\right]^2 = \frac{\varepsilon_1^2 - \varepsilon_3^2 - \varepsilon_2(2\varepsilon_1 - 2\varepsilon_3)}{4} \tag{5.7}$$

which is independent of both frequency and wave number, and, unlike (5.6) gives a fixed conic section on the E_0^2 / E_b^2 plane for particular values of permittivities. In contrast, equation (5.6) is dependent on frequency and wave number as well as the permittivities and values of the intensities at the boundaries.

If the E_0 used in the dispersion equation is set to the value $E_0^{(l)}$ and a point ϖ and κ on the dispersion curve is taken the boundary field relationship yields a unique hyperbola. Four cases are important. These are

$$\kappa_2^2 > 0, \quad C_2 \geq 0, \quad \kappa_2^2 < 0, \quad C_2 \leq 0, \quad \kappa_2^2 < 0, \quad C_2 < 0 \tag{5.8}$$

For example, a surface wave of order 1 is given when $\kappa_2^2 > 0, C_2 \geq 0$ the electric field in the film is

$$E_2 = b\,\mathrm{sc}\left(\mathrm{sc}^{-1}\left[\frac{E_0}{b}\bigg|m\right] - a\varpi|\mu|^{\frac{1}{2}} Z\right) \tag{5.9}$$

where

$$a^2 = \kappa_2^2 + \sqrt{\kappa_2^4 - 4\varpi^2|\mu|C_2}\,/2\varpi^2|\mu| \quad > \quad b^2 = \kappa_2^2 - \sqrt{\kappa_2^4 - 4\varpi^2|\mu|C_2}\,/2\varpi^2|\mu| \quad m = (a^2 - b^2)/a^2 \tag{5.10}$$

and a surface wave of order zero when $\kappa_2^2 > 0, C_2 < 0$, with an electric field given by

$$E_2 = \sqrt{a^2 + b^2}\,\mathrm{ds}\left(\sqrt{a^2 + b^2}\,\varpi|\mu|^{\frac{1}{2}} Z + \mathrm{ds}^{-1}\left[E_0\big/\sqrt{a^2 + b^2}\right]\right) \tag{5.11}$$

where

$$a^2 = \left(\kappa_2^2 + \sqrt{\kappa_2^4 + 4\varpi^2 |\mu||C_2|}\right)\Big/2\varpi^2|\mu| \quad b^2 = \left(\sqrt{\kappa_2^4 + 4\varpi^2 |\mu||C_2|} - \kappa_2^2\right)\Big/2\varpi^2|\mu| \quad m = a^2\Big/\left(a^2 + b^2\right) (5.12)$$

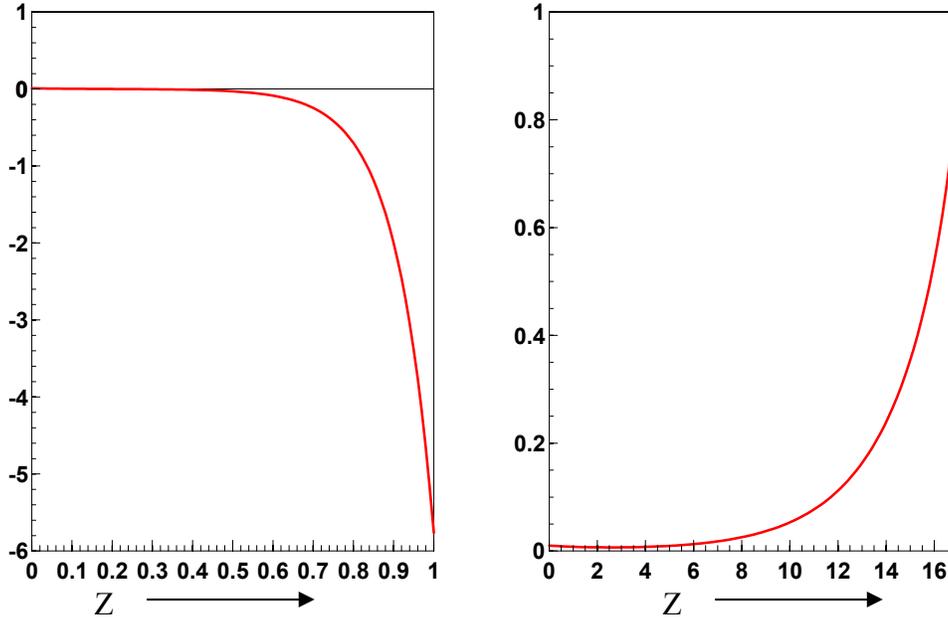Fig. 8. shows examples of these two wave profiles across the film.

Fig. 8. The field profile across the film for (a) an order 1 solution and (b) an order 0 solution respectively.

For (a) $\varpi = 0.47$, $\kappa_x = 10.498$, $E_0^{(1)} = 0.01$, $D = 1$, $\varepsilon_1 = 2,45$, $\varepsilon_3 = 1$ and

for (b) $\varpi = 0.0.485$, $\kappa_x = 0.84465$, $E_0^{(1)} = 0.01$, $D = 17$, $\varepsilon_1 = 2,45$, $\varepsilon_3 = 1$.

These cases show the production of unique surface waves due to a combination of nonlinear and metamaterial properties. They can only exist because of the metamaterial properties, but over some frequency range it may be possible to achieve similar field distributions in the linear regime. They are surface waves that are unique to this kind of material and could not occur if the permeability is set equal to unity.

7. CONCLUSIONS

This paper ranges over a number of topics, all of which are associated with soliton propagation or a new departure into the area of strongly nonlinear guided waves. An attempt has been made to indicate previous progress towards setting up the fundamental nonlinear Schrodinger equation. The development follows the main emphasis in the literature which is on third-order nonlinearity. No attempt has been made to discuss second-harmonic generation. This has been deferred for later publications. After setting up the basic theory and emphasizing some misunderstanding in the literature, the basic nonlinear Schrödinger equation is modified to take into account a special form of diffraction-management. This is followed up by a traditional modulation instability analysis and the outcome is supported by detailed soliton formation simulations. The latter include a novel, but very important, process called nonlinear diffraction, and the examples given tend to focus upon nonlinear couplers. After all this work on weakly nonlinear systems that rely upon a slowly varying envelope approach in which the linear modal field remains untouched by the nonlinearity, a new direction is developed that involves strongly nonlinear waves. The latter are the outcomes of an exact solution to Maxwell's equations and are so far away from the slowly varying approach, that there is no question that unmodified, linear modal fields will be a feature of the solutions. In fact, the guided modes of a nonlinear film of metamaterial are substantially altered by the power, and it is shown above that a unique concentration of guided energy is created at a guide surface. This is a creation of a surface guided mode that is a direct consequence of the power flow coupling to the change in the boundary

conditions from the PPM case that is brought about by the metamaterial properties. Finally, the paper shows that nonlinear tunability of negative phase metamaterials is a fascinating area that promises many potential applications.

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